FRIEDRICH-SCHILLER-UNIVERSITÄT JENA

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Stochastic Analysis (SS 2019)

Exercise sheet 7

Submission: 28.05.2019

Problem 1 **(6 Points)**

Let $(M_t)_{t\geq 0}$ be a continuous martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

(a) Show that if $M_t - M_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ for all $t \geq 0$, then

$$\mathbb{E}\big[(M_t - M_0)^2\big] = \mathbb{E}\left[\sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})^2\right]$$

for arbitary $0 = t_0 < t_1 < \ldots < t_n = t$, $n \in \mathbb{N}$.

(b) Show that if all sample paths of M have a finite and even uniformly bounded (total) variation, then almost all paths of M are constant functions.

(c) Show that if for all $\omega \in \Omega$ and all $t > 0$ the sample path $s \mapsto M_s(\omega)$ has a finite variation on $[0, t]$, then almost all paths of M are constant functions.

Problem 2 **(4 Points)**

Let $(B_s)_{s\geq 0}$ be a Brownian motion.

(a) Let $f(s) := (1+s)^{-\frac{3}{2}}$, $s \in [0, \infty)$, and $M_t := \int_0^t f(s)\,dB_s$, $t \in [0, \infty)$. Show that $M \in H_0^2$ is well-defined and compute its norm $||M||_{\mathbb{H}_2}$.

(b) Determine all $\alpha \in (0, \infty)$ for which $M_t^\alpha := \int_0^t (1+s)^{-\alpha}\,dB_s$, $t \geq 0$, is well-defined as an L^2-bounded martingale.

Problem 3 - Euler scheme **(5 Points)**

Let $(B_t)_{t\geq 0}$ be a Brownian motion. Let $f:[0,\infty)\to\mathbb{R}$ be Lipschitz continuous with Lipschitz constant L_f. For any $n\in\mathbb{N}$ define

$$\begin{aligned} f^n(s) &= \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \mathbb{1}_{\left[\frac{k}{n},\frac{k+1}{n}\right)}(s), \qquad k\in\mathbb{Z}_{\geq 0}, \\ M_t^n &= \int_0^t f^n(s)\,dB_s, \\ \text{and }\ M_t &= \int_0^t f(s)\,dB_s. \end{aligned}$$

Write M_t^n more explicitly as a linear combination of increments of the Brownian motion and show that for $T>0$ we have

$$\mathbb{E}\big[\,|M_T - M_T^n|\,\big] \leq \frac{C(T,L_f)}{n},$$

with some constant $C(T,L_f)$ that depends only on T and L_f and is non-decreasing in these values.

Total: 15 Points

Terms of submission:

- Solutions can be submitted in groups of at most 2 students.
- Please submit at the beginning of the lecture or until 9:50 a.m. in room 3523, Ernst-Abbe-Platz 2.